A Note on N!

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To consider *all* defined terms (notation, words, etc.) as abbreviations, with a definition as the statement indicating the abbreviation and the language it is to abbreviate, would seem to be a more general and simple approach to the idea of definitions.

There are certain unwritten rules in the use of mathematical terms and definitions. During the course of development of mathematical language, certain words and symbols have been accepted in a particular mathematical context to have somewhat standard definitions. That is, the same term is used by all to replace somewhat the same language. If a mathematician uses such a familiar term in the common mainstream manner, then it is not always necessary to exhibit the definition. However, it is one's right to use any term in any manner as long as communication is not sacrificed. If one chooses to use a familiar term but in a different manner than would be expected, a course of action which is permissible though not advisable, then the person is obligated for the sake of clarity to explicitly reveal a definition.

Another existing situation, possibly unfortunate, is that certain terms, especially symbols, have several standard definitions. To know which definition to consider depends on the context in which the term is used and left for the judgment of the reader. For example, "$-$" may appear as part of a name for a negative number as "-2"; to indicate the image (difference) of two numbers under the binary operation (function) subtraction as "$6-3$"; to indicate the additive inverse (opposite) of a number as "$-(+2)$"; or the difference of sets as "$A-B$." However, when "$-$" appears in a piece of language it is assumed that the reader would consider the definition for "$-$" which makes the language meaningful. As in the previous examples, it would be clear that the "$-$" in "$-(+2)$" does not mean difference, for there is no indication of a minuend. But there are situations that can be ambiguous. For instance, if the convention is used where "5" abbreviates "$+5$," then the language "-5" is meaningful to consider the "$-$" as either part of the standard numeral for the negative number or as indicating the additive inverse of the number $+5$. In this case, however, and I might add "thank heavens," either meaningful interpretation refers to the same number. That is, negative five is the additive inverse of positive five. Ambiguity usually leads to more disastrous results.

A NOTE ON N!

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In this note two theorems are proved. The first states:

THEOREM 1. *If A is any positive integer having m digits, there exists a positive integer N such that the first m digits of $N!$ constitute the integer A.*

In order to prove this theorem the following two lemmas are needed.

LEMMA 1. *The fractional part of* $\log N$, *written* $\{\log N\}$ *is dense on the unit interval.*

Proof. $\{\text{Log}_{10} x\}$ determines all of the digits of x and the integral part determines only the location of the decimal point. The terminating decimals are dense in $[1, 10]$ and $\log x = \{\log_{10} x\}$, $1 \leqq x \leqq 10$ is continuous. Therefore $\{\log_{10} x\}$ is dense on $[0, 1]$. Since $\log_a(x) = \log_b(x) \log_a b$, the lemma follows for any base.

LEMMA 2. *$\{\log N!\}$ is dense on* $[0, 1]$.

Proof.

$$\begin{aligned}\log(N+k)! &= \log N! + \sum_{j=1}^{k} \log\left(1 + \frac{j}{N}\right) N \\ &= \log N! + k \log N + \sum_{j=1}^{k} \log\left(1 + \frac{j}{N}\right) \\ &= \log N! + k \log N + \sum_{j=1}^{k}\left(\frac{j}{N} + O\left(\frac{j^2}{2N^2}\right)\right) \\ &= \log N! + k \log N + \frac{k}{2N}(k+1) + O\left(\frac{k^4}{N^2}\right).\end{aligned}$$

We shall now construct an η-net on $[0, 1]$ with elements of $\{\log N!\}$ where η is small.

Let $\eta > 0$. Since $\{\log N\}$ is dense on $[0, 1]$, there is an infinite set of N, say N_α, such that $5\eta/8 < \{\log N_\alpha\} < 7\eta/8$. Let $M - 1 = [16/9\eta]$, the greatest integer in $16/9\eta$. Choose N_α so large that

$$\left|\frac{M(M+1)}{2N_\alpha} + O\left(\frac{M^4}{N_\alpha^2}\right)\right| < \frac{\eta}{16}.$$

Then

$$\{\log(N_\alpha + k)!\} = \left\{\log N_\alpha! + k \log N_\alpha + \frac{k(k+1)}{2N_\alpha} + O\left(\frac{k^4}{N_\alpha^2}\right)\right\} = g(k).$$

Since N_α is fixed, $\log N_\alpha$ and $\log N_\alpha!$ are fixed. For $k \leqq M$

$$\left|\frac{k(k+1)}{2N_\alpha} + O\left(\frac{k^4}{N_\alpha^2}\right)\right| < \frac{\eta}{16}.$$

Therefore the principal part of the variation of $g(k)$ as k varies is due to the term $k \log N_\alpha$. If $g(k) < 1 - \eta$, and since $\{\log N_\alpha\} < 7\eta/8$, then

$$\begin{aligned}g(k+1) - g(k) &< \{\log N_\alpha\} + 2\left|\frac{M(M+1)}{2N_\alpha} + O\left(\frac{M^4}{N_\alpha^2}\right)\right| \\ &< 7\eta/8 + \eta/8 = \eta.\end{aligned}$$

If $g(k) > 1 - (1/2)\eta$, since $\{\log(N_\alpha)\} > (5/8)\eta$, it follows that $g(k+1) > (7/8)\eta + (1/8)\eta = \eta$. Since $M\{\log N_\alpha\} > (16/9\eta + 1)(5/8)\eta > (16/9\eta)(5/8)\eta = 10/9$, the whole unit interval is covered by an η net of points of $\{\log N!\}$ where the chosen N are $N_\alpha, N_\alpha+1 \cdots N_\alpha+M$.

Proof of Theorem 1. If we take log to the base 10, we want there to exist an integer t and an integer N such that $A\times 10^t \leqq N! < (A+1)10^t$ or that $t+\log A \leqq \log N! < t+\log(A+1)$. This can be done if

$$\{\log A\} < \{\log N!\} < \{\log(A+1)\}.$$

By Lemma 2 $\{\log N!\}$ is dense in $[0, 1]$. Thus, such an N can be chosen and the theorem is proved.

Theorem 2 is a generalization of Lemma 2. We need the following definitions and lemma.

Define $\log^{(j)} m$ *to be the jth iterant of* $\log m$, *i.e.*, $\log^{(2)} m = \log\log m$.

Define $\eta!^{(j)}$ *to be the jth iterant of* $n!$, *i.e.*, $n!^{(2)} = n!!$

LEMMA 3. *For* $j > 1$,

$$\log^{(j)}(n+k)!^{(j)} = \log n! + \log\log n! + k\log n + O\left(\frac{k\log n}{\log n!}\right).$$

Proof. By Stirling's formula,

$$\begin{aligned}\log^{(2)}(n+k)!^{(2)} &= \log([(n+k)!+1/2]\log(n+k)! - (n+k)!\log e + O(1)) \\ &= \log[(n+k)!\log(n+k)!]\left[1+O\left(\frac{1}{\log(n+k)!}\right)\right] \\ &= \log(n+k)! + \log\log(n+k)! + O\left(\frac{1}{\log(n+k)!}\right).\end{aligned}$$

From the proof of Lemma 2, $\log^{(2)}(n+k)!^{(2)}$

$$\begin{aligned}&= \log n! + k\log n + \log[\log n! + k\log n] + O(k^2/n^2) \\ &= \log n! + \log\log n! + k\log n + O\left(\frac{k\log n}{\log n!}\right),\end{aligned}$$

establishing a basis.

Induction hypothesis:

$$\log^{(j)}(n+k)!^{(j)} = \log n! + \log\log n! + k\log n + O\left(\frac{k\log n}{\log n!}\right).$$

Then by Stirling's formula,

$$\begin{aligned}&\log^{(j+1)}(n+k)!^{(j+1)} \\ &\quad = \log^{(j)}([(n+k)!^{(j)}+1/2]\log(n+k)!^{(j)} - (n+k)!^{(j)}\log e + O(1)) \\ &\quad = \log^{(j)}\left[(n+k)!^{(j)}\left[1+1/2\,\frac{\log(n+k)!^{(j)}}{(n+k)!^{(j)}} + O\left(\frac{1}{(n+k)!^{(j)}}\right)\right]\right]\end{aligned}$$

$$= \log^{(j-1)}\left(\log(n+k)!^{(j)} + 1/2\,\frac{\log(n+k)!^{(j)}}{(n+k)!^{(j)}} + O\left(\left(\frac{\log(n+k)!^{(j)}}{(n+k)!^{(j)}}\right)^2\right)\right)$$

$$= \log^{(j-1)}\left[\log(n+j)!^{(j)}\left[1 + 1/2\,\frac{1}{(n+k)!^{(j)}} + O\left(\frac{\log(n+k)!^{(j)}}{[(n+k)!^{(j)}]^3}\right)\right]\right]$$

$$= \log^{(j)}(n+k)!^{(j)} + O\left(\frac{1}{(n+k)!^{(j)}}\right)$$

$$= \log n! + \log\log n! + k\log n + O\left(\frac{k\log n}{\log n!}\right),$$

completing the proof.

From this, as in Lemma 2, one can construct an η net and prove:

THEOREM 2. *$\{\log^{(j)} N!^{(j)}\}$ is dense on the unit interval.*

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A CATEGORICAL SYSTEM OF AXIOMS FOR THE COMPLEX NUMBERS

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In many elementary analysis texts the real numbers are introduced by means of a categorical system of axioms. In contrast one finds in complex analysis texts that the complex numbers are defined by means of some constructive process. The purpose of this note is to remedy this omission by giving a brief exposition of the complex numbers starting from a categorical system of axioms. In this paper it is assumed that the elementary properties of a complete linearly ordered field are known.

DEFINITION 1. *A complex number system K is defined to be any commutative field K with the following property: There exists a homomorphism $T: K \to K$ such that*

(1) $T(z) \neq z$ *for at least one* $z \in K$,

(2) $T(T(z)) = z$ *for all* $z \in K$,

(3) *the subset* $R = \{z \in K: T(z) = z\}$ *is a complete linearly ordered field.*

The mapping T is called the conjugate operator. For any element $z \in K$, $T(z)$ is called the conjugate of z and we denote it by $\bar{z}$. The elements z in K are called complex numbers.

THEOREM 1. *For any complex number system K, the following properties of the conjugate operator are valid:*

(4) $\bar{z} \neq z$, *for at least one* $z \in K$,

(5) $\bar{\bar{z}} = z$, *for all* $z \in K$,