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A Note on an Identity of Ramanujan

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In a forthcoming paper **[1]**, Berndt and Bhargava have supplied a proof of this eye-catching identity of Ramanujan found in his third notebook [**3**, p. 386]: if $ad = bc$, then

$$\begin{aligned}64\{(b+c+d)^6 - (a+c+d)^6 - (a+b+d)^6 + (a+b+c)^6 \\ + (a-d)^6 - (b-c)^6\} \\ \times\{(b+c+d)^{10} - (a+c+d)^{10} - (a+b+d)^{10} \\ + (a+b+c)^{10} + (a-d)^{10} - (b-c)^{10}\} \\ = 45\{(b+c+d)^8 - (a+c+d)^8 - (a+b+d)^8 \\ + (a+b+c)^8 + (a-d)^8 - (b-c)^8\}^2.\end{aligned}$$

It figures also in their expository article **[2]** featuring a selected group of Ramanujan's results. Unfortunately, they have missed its simple proof and so its genesis by not noticing that it is built from two sets of sums:

$$u_n = \alpha_1^n + \beta_1^n + \gamma_1^n, \quad \alpha_1 = b + c + d, \quad \beta_1 = -(a + b + c), \quad \gamma_1 = a - d,$$

$$v_n = \alpha_2^n - \beta_2^n + \gamma_2^n, \quad \alpha_2 = a + c + d, \quad \beta_2 = -(a + b + d), \quad \gamma_2 = b - c.$$

By $\alpha_j + \beta_j + \gamma_j = 0$, the underlying problem is to compute

$$\omega_n = \alpha^n + \beta^n + \gamma^n,$$

where α, β and γ are the roots of the cubic

$$z^3 - pz + q = 0.$$

It is simple to work out an easy special case of Newton's formulae for power sums of the roots of an algebraic equation. Indeed, the obvious recursion

$$\omega_{n+3} - p\omega_{n+1} + q\omega_n = 0$$

with the initial values

$$\omega_{-1} = \frac{p}{q}, \qquad \omega_0 = 3, \qquad \omega_1 = 0,$$

yields

$$\omega_2 = 2p, \qquad \omega_4 = 2p^2,$$

$$\omega_3 = -3q, \qquad \omega_5 = 5pq, \qquad \omega_7 = -7p^2q,$$

$$\omega_6 = 2p^3 + 3q^2, \qquad \omega_8 = 2p^4 + 8pq^2, \qquad \omega_{10} = 2p^5 + 15p^2q^2.$$

Form the cubic whose roots are α_j, β_j and γ_j:

$$z^3 - p_j z + q_j = 0.$$

We have

$$p_1 = (b + c + d)(a + b + c) + (a - d)^2,$$

$$p_2 = (a + c + d)(a + b + d) + (b - c)^2,$$

$$p_1 - p_2 = 3(bc - ad).$$

Hence $p_1 = p_2$ if and only if

$$ad = bc.$$

Assume this condition and set

$$p_1 = p_2 = P, \qquad \Delta = q_1^2 - q_2^2.$$

Now the $u_n = \omega_n(p_1, q_1)$ and the $v_n = \omega_n(p_2, q_2)$ given by the computed $\omega_n = \omega_n(p, q)$ show that

$$u_2 = v_2, \qquad u_4 = v_4,$$

$$u_6 - v_6 = -3\Delta, \qquad u_8 - v_8 = 8P\Delta, \qquad u_{10} - v_{10} = 15P^2\Delta.$$

So we have Ramanujan's ingenious parametric construction of equal sums of three nth powers ($n = 2, 4$), and Ramanujan's identity. Clearly, for both these results, the condition $ad = bc$ is crucial. Ramanujan must have been primarily looking for the first one because of its number-theoretic signifiance, the second being incidental and apparently the only one of its kind in this context.

For special choices of the parameters, the equal sums of three nth powers ($n = 2, 4$) constructed by Ramanujan may present the same terms! This happens, for instance, when

$$a = b\ (c = d), \qquad a = c\ (b = d), \qquad b = 0 = d\ (a \neq 0), \qquad c = 0 = d\ (a \neq 0).$$

Barring such cases, the construction yields numbers expressible as sums of three nth powers ($n = 2, 4$) in two different ways. This observation, which we owe to a comment of the referee/editor, does not point to any flaw in the construction for which what really matters is its *algebraic* formulation.

I wish to thank Professor Bhargava for having kindly shown me the proof sheets of [1] and a preprint of [2].

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On an Identity of Daubechies

Doron Zeilberger

Tossing a coin (whose $Pr(head) = p$) until reaching n heads or n tails and equating the probability, 1, of finishing with the sum of the probabilities of all the possible final outcomes leads to

$$\sum_{i=0}^{n-1} \binom{n+i-1}{i} p^n (1-p)^i + \sum_{i=0}^{n-1} \binom{n+i-1}{i} p^i (1-p)^n = 1,$$

which was proved in [1], (pp. 167–171) and [2] using Bezout's theorem and induction respectively. Rolling a k-faced die instead leads to the multivariate generalization

$$\sum_{i=1}^{k} \sum_{\substack{0 \le \alpha_j \le n-1 \\ j \ne i}} \frac{(\alpha_1 + \cdots + \alpha_{i-1} + (n-1) + \alpha_{i+1} + \cdots + \alpha_k)!}{\alpha_1! \ldots \alpha_{i-1}!(n-1)!\alpha_{i+1}! \ldots \alpha_k!} \times$$

$$p_1^{\alpha_1} \cdots p_{i-1}^{\alpha_{i-1}} p_i^n p_{i+1}^{\alpha_{i+1}} \cdots p_k^{\alpha_k} = 1,$$

provided $p_1 + \cdots + p_k = 1$.

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