On Some Irrational Decimal Fractions

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It is known that the decimal fraction

$$\alpha = 0.235711131719\ldots$$

is irrational, where the sequence of digits is formed by the primes in ascending order. In [**1**, Th. 138] there are two different proofs for this statement. The first uses a special case of the Dirichlet's theorem, namely: any arithmetical progression of the form $10^{s+1}k + 1$ $(k = 1, 2, \cdots)$ contains primes. In the second proof it is assumed that there is a prime between N and $10N$ for every $N > 0$, which is the special case of the Bertrand's Postulate. Similar proofs are found in [**2**].

In this article we will give a direct proof for this statement. We prove even more.

Theorem. *Let $1 \leqslant a_1 < a_2 < \ldots$ be a sequence of integers for which $\sum_{i=1}^{\infty} 1/a_i = \infty$. Then the decimal fraction $\alpha = 0 \cdot (a_1)(a_2)\ldots(a_n)\ldots$ is irrational.*

Since $\sum_{i=1}^{\infty} 1/p_i = \infty$, where $p_1 < p_2 < \ldots$ is the sequence of primes, we immediately get the original version of the statement.

Definition. Let B be a block of digits $b_1 b_2 \ldots b_s$ with $s \geq 1$ and $0 \leq b_i \leq 9$ for $i = 1, 2, \ldots, s$. Let n be a positive integer $\sum_{i=0}^{k} c_i 10^{k-i}$ with $c_0 \neq 0$. The integer n is said to contain the block of digits B if for some $j \geq 0$ we have $c_{i+j} = b_i$ for every $i = 1, 2, \ldots, s$. For example, the integer 1402857 contains the blocks 14 and 0285 (among others), but not the blocks 014 or 582.

Lemma. *If $X = X(b_1, b_2, \ldots, b_s)$ denotes the sequence of positive integers not containing the block of digits $b_1 b_2 \ldots b_s$, then $\sum_{n \in X}^{\infty} 1/n$ is convergent.*

We mention that the Lemma is a generalization of a well-known exercise (see [**1**, Th 144]).

Proof of the Lemma: Let $s_n = 1/x_1 + 1/x_2 + \ldots 1/x_n$ and let t be an integer for which $x_{t-1} < 10^s \leq x_t$. Then we have

$$s_n < 1/x_1 + 1/x_2 + \ldots + 1/x_t + 10^{-s}\left(1/[x_{t+1}/10^s] + \ldots + 1/[x_n/10^s]\right).$$

We note that if $t < i \leq n$, then $[x_i/10^s]$ is a member of X, say x_j. Also, since the block $b_1 b_2 \ldots b_s$ appears in at least one of 10^s consecutive integers, it follows that for any fixed x_j there are at most $10^s - 1$ values of x_i such that $[x_i/10^s] = x_j$, and we have

$$s_n < \sum_{i=1}^{t} 1/x_i + (10^s - 1)10^{-s} s_n \quad \text{or} \quad s_n < 10^s \cdot \sum_{i=1}^{t} x_i,$$

which proves the lemma.

Proof of the Theorem: Assume that α is a rational number. Thus α is a periodic decimal, with a block of digits, say $b_1 b_2 \dots b_s$, repeating endlessly perhaps after an initial first block. If B is a block of 1's, define $c_1 c_2 \dots c_{2s}$ to be a block of 2's of length $2s$; otherwise define $c_1 c_2 \dots c_{2s}$ to be a block of 1's of length $2s$. Now define $Y = Y(c_1, c_2, \dots, c_{2s})$ as the sequence of natural numbers not containing the block of digits $c_1 c_2 \dots c_{2s}$. If we write

$$\sum_{i=1}^{\infty} 1/a_i = \sum_{a \in Y} 1/a + \sum_{a \notin Y} 1/a,$$

then by the Lemma the first sum on the right side converges, and hence the second sum diverges. This implies that there are infinitely many a_i that contain the block of digits $c_1 c_2 \dots c_{2s}$. This in turn implies that B cannot be a repeating block of digits in α. This contradiction establishes the Theorem.

ACKNOWLEDGMENT. The author would like to thank the referee for a number of suggestions and for detecting some flaws in our original version.

REFERENCES

1. Hardy–Wright, *An Introduction to the Theory of Numbers*, fifth edition, Oxford, Clarendon Press, 1979.
2. G. Pólya–G. Szegö, *Problems and Theorems in Analysis II.*, Springer-Verlag, 1976, (exercise 257.)

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Professor Florian Cajori died suddenly of pneumonia on August 14, 1930, at his home in Berkeley, California. He was a charter member of the Mathematical Association of America and was one of an original group of four (later enlarged to twelve) representatives of mid-western universities and colleges who made possible the re-establishment of the American Mathematical Monthly on a sound financial basis. A detailed account of his historical researches will be published in the *Monthly* in due course.

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