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A Simple Direct Proof of Marden's Theorem

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Abstract. Marden's theorem characterizes the critical points of complex polynomials of degree 3 in a nice geometrical way. Our proof of the theorem is based directly on the defining property of ellipses.

"Marden's theorem" (proven much earlier by J. Siebeck; see [**1**], [**2**] and the references cited there) states that the critical points $e, f \in \mathbb{C}$ of a complex polynomial p of degree 3,

$$p(z) = (z-a)(z-b)(z-c) = z^3 - (a+b+c)z^2 + \cdots$$
$$p'(z) = 3(z-e)(z-f) = 3\left(z^2 - (e+f)z + ef\right)$$

are the *foci of the Steiner inellipse* of the triangle with vertices $a, b, c \in \mathbb{C}$:

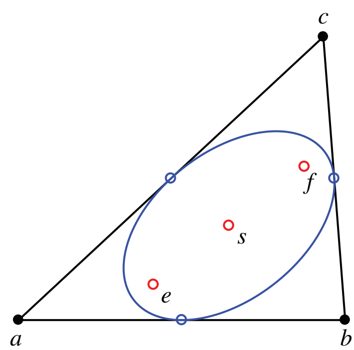

Figure 1. Steiner inellipse with center and foci

The *Steiner inellipse* of a triangle abc is the image of the incircle of an equilateral triangle $a_0b_0c_0$, under the affine transformation that maps a_0 to a, b_0 to b, and c_0 to c. It is tangent to each side of the triangle at its midpoint and its center is the centroid s of the triangle.

The Steiner inellipse is the unique ellipse with center s that passes through all three midpoints of the sides of the triangle abc (the *Steiner circumellipse* of the medial triangle). Indeed, in the case of a circle, this is possible only if the triangle is circumscribed and equilateral.

Proof. To prove Marden's theorem we may assume that the point $0 \in \mathbb{C}$ is the triangle's centroid s:

$$a + b + c = 0 \quad \text{and thus} \quad e + f = 0. \tag{1}$$

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In this case, the derivative of p,

$$p'(z) = (z-a)(z-b) + (z-a)(z-c) + (z-b)(z-c) \tag{2}$$

$$= (z-a)(z-b) + \big(2z-(a+b)\big)(z-c) \tag{3}$$

can also be written as

$$p'(z) = 3(z+e)(z-e). \tag{4}$$

For the triangle's side midpoint $z_1 = \frac{a+b}{2}$, we find from formulas (4) and (3)

$$3(z_1+e)(z_1-e) = -\left(\frac{a-b}{2}\right)^2. \tag{5}$$

For the sum of the distances from z_1 to the points $-e$ and e, the *parallelogram identity*, formula (5), and formula (1) ($a+b=-c$) yield

$$\begin{aligned} 2\left(|z_1+e|+|z_1-e|\right)^2 &= 2|z_1+e|^2 + 2|z_1-e|^2 + 4|(z_1+e)(z_1-e)| \\ &= 4|z_1|^2 + 4|e|^2 + \tfrac{1}{3}|a-b|^2 \\ &= |a+b|^2 + 4|e|^2 + \tfrac{1}{3}|a-b|^2 \\ &= \tfrac{1}{3}\left(|a+b|^2 + |a-b|^2\right) + \tfrac{2}{3}|a+b|^2 + 4|e|^2 \\ &= \tfrac{2}{3}\left(|a|^2+|b|^2+|c|^2\right) + 4|e|^2. \end{aligned}$$

But this last sum is independent of our choice of the triangle's side midpoint! The *ellipse E with foci $-e$ and e (and center 0) through the midpoint* $\frac{a+b}{2}$ thus also passes through the midpoints $\frac{a+c}{2}$ and $\frac{b+c}{2}$ and therefore *is* the Steiner inellipse of the triangle abc. ■

From formula (5) we also might obtain a self-contained complex proof of Marden's theorem (without referring to affine transformations). By considering *arguments* in addition to *absolute values*, $\arg(z_1+e)+\arg(z_1-e) = -2\arg(a-b)$, we see that a light beam from the focus $-e$ of E, reflected at the triangle's side ab in z_1, then passes through the focus $+e$. The ellipse E is thus tangent to the side ab at its midpoint z_1. Since E passes through the other midpoints as well and since we may name the triangle's vertices arbitrarily, E is tangent to every side of the triangle at its midpoint.

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