NOTES

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Inverse Functions of $y = x^{1/x}$

Yunhi Cho and Kyunghwan Park

1. INTRODUCTION. For $y > 0$, consider the problem of solving $x = \log_y x$, i.e., of finding all numbers x that are their own logarithms (note that x must be positive for the right-hand side to be defined). This problem is equivalent to solving for x the exponential equation $y^x = x$, or $y = x^{1/x}$. Hence the solutions x to $x = \log_y x$ belong to the inverse relation of the function $y = x^{1/x}$. By elementary calculus one sees that the function $y = x^{1/x} = e^{\ln(x)/x}$ is increasing on $(0, e]$ with image $(0, e^{1/e}]$ and decreasing on $[e, \infty)$ with image $(1, e^{1/e}]$ (see Figure 1). Thus to find all solutions to our original problem, it suffices to find the inverse function for $y = x^{1/x}$, first on the interval $(0, e]$, and then on the interval $[e, \infty)$. For the actual computations, it is convenient to divide the inverse relation or multi-function into three branch functions, the inverse of $y = x^{1/x}$ restricted first to $(0, 1/e]$, then to $[1/e, e]$, and finally restricted to $[e, \infty)$.

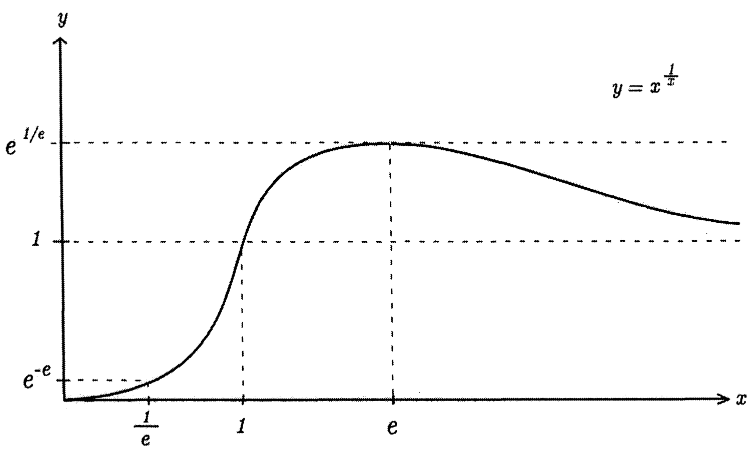

Figure 1. $y = x^{\frac{1}{x}}$

Consider the function, usually called the hyperpower function, defined as the sequential power limit $x = y^{y^{y^{\cdot^{\cdot^{\cdot}}}}}$. This function was discovered by Euler and has been rediscovered and studied by other mathematicians; see [**2**] for some of its history, properties, and other references. It converges for y in the interval $[e^{-e}, e^{1/e}]$, diverges for positive y outside this interval, and satisfies the equation $x = y^x$ on this interval (see [**2**], [**3**]). Hence it is the inverse of $y = x^{1/x}$ restricted to the interval $[1/e, e]$. Our object in this article is to find representations of the inverse of $y = x^{1/x}$ restricted to $(0, 1/e]$ and restricted to $[e, \infty)$ and use these representations to study the inverses of functions such as $y = x^x$, $y = xe^x$, $y = x \ln x$, and $y = x + e^x$.

2. CHARACTERIZATION OF INVERSE FUNCTIONS OF $y = x^{1/x}$.

The following is our main result:

Theorem 1. *The inverse branch functions of $y = x^{1/x}$ are represented by*

(a) $x = y^{y^{y^{\cdot^{\cdot^{\cdot}}}}}$, $(1/e \le x \le e, e^{-e} \le y \le e^{1/e})$ *and*
(b) $x = \cdots \log_y \log_y e$, $(e \le x, 1 < y \le e^{1/e})$ *and*
(c) $x = \cdots \log_y \log_y(1/e)$, $(0 < x \le 1/e, 0 < y \le e^{-e})$.

Furthermore, for each of the cases (a), (b), and (c), the sequential limit does not exist for positive y outside the specified region.

Proof. Since the hyperpower function $x = y^{y^{y^{\cdot^{\cdot^{\cdot}}}}}$ is the inverse function on the region given in part (a) of the theorem, and it diverges for $y > 0$ not in the interval $[e^{-e}, e^{1/e}]$, we consider the other two cases.

To prove (b), consider the graph of $y = x^{1/x}$ beyond $x = e$. This function is decreasing on $[e, \infty)$, and hence there exists for each $y \in (1, e^{1/e}]$, the range, a unique $x \in [e, \infty)$ such that $y = x^{1/x}$. Moreover, $y = x^{1/x}$ is equivalent to $\log_y x = x$, and thus the latter has exactly one solution $\overline{x}$ with $\overline{x} \ge e$. The fixed point equation $\log_y x = x$ motivates the choice of the iterating function $f(x) := \log_y(x) = \ln x / \ln y$, and we consider the sequence obtained by iterating:

$$x_n := f^n(e), \quad n \ge 0.$$

Since $y \le e^{1/e}$, applying the order-preserving function f to both sides we obtain $e \le f(e)$, and thus

$$e \le f(e) \le f^2(e) \le \cdots,$$

an increasing sequence. In addition, since $e \le \overline{x}$, then $f(e) \le f(\overline{x}) = \overline{x}$; it follows by induction that $x_n \le \overline{x}$ for every x_n. Thus the bounded monotone sequence x_n converges to some real number x above e, and by continuity of f, $f(x) = x$. Since the fixed point is unique, it follows that $\overline{x} = x = \lim_{n\to\infty}(\log_y)^n(e)$.

Now if $y > e^{1/e}$, then y is not in the range of $y = x^{1/x}$, and thus the equation $\log_y x = x$ has no solution. Thus the limit $x = \lim_{n\to\infty}(\log_y)^n(e)$ cannot exist, for otherwise by continuity, $\log_y x = x$. On the other hand, if $0 < y < 1$, then $\log_y e = \ln e / \ln y = 1/\ln y < 0$, and thus $\log_y \log_y e$ is not even defined, and hence the sequence is not defined.

To prove (c), assume that $0 < y < e^{-e}$. Note that

$$0 < y < e^{-e} \Leftrightarrow \ln y < -e \Leftrightarrow \frac{1}{-\ln y} < \frac{1}{e}.$$

Since $f(x) = \log_y x = \ln x / \ln y$ is strictly decreasing, hence order-reversing, we have $0 < f(1/e) = -1/\ln y = 1/(-\ln y) < 1/e$. Applying the order-reversing f to the ends of the inequality yields $f^2(1/e) > f(1/e)$. In addition, by the Mean Value Theorem, $f^2(1/e) - f(1/e) = f'(c)(f(1/e) - (1/e))$ for some c, $f(1/e) < c < 1/e$. Since

$$|f'(c)| = \left|\frac{1}{c \ln y}\right| = \frac{1}{-\ln y} \cdot \frac{1}{c} = f\left(\frac{1}{e}\right) \cdot \frac{1}{c} < c \cdot \frac{1}{c} = 1,$$

we conclude that $f^2(1/e) - f(1/e) < (1/e) - f(1/e)$, i.e., $f^2(1/e) < 1/e$. By the order-reversing property of f and induction, we obtain

$$0 < f\left(\frac{1}{e}\right) < f^3\left(\frac{1}{e}\right) < f^5\left(\frac{1}{e}\right) < \cdots, \quad \cdots < f^4\left(\frac{1}{e}\right) < f^2\left(\frac{1}{e}\right) < \frac{1}{e},$$

and $f^{2m+1}(1/e) < f^{2n}(1/e)$ for all non-negative integers m, n. The monotonically decreasing bounded sequence of even powers converges to some b, the monotonically increasing bounded sequence of odd powers converges to some $a \leq b$, and by continuity of f, $f(a) = b$ and $f(b) = a$. If $a < b$, we can again apply the Mean Value Theorem to the interval $[a, b]$ and conclude that $f(a) - f(b) < b - a$, a contradiction. Thus $a = b$ is a fixed point for $f(x) = \log_y x$, hence must be the unique fixed point, and $a = \lim_{n\to\infty}(\log_y)^n(1/e)$.

Now if $y \geq 1$, then $\log_y(1/e) = -1/\ln y < 0$, and thus $\log_y \log_y(1/e)$ is not defined. For the case $e^{-e} < y < 1$, let us suppose that the sequence $\{f^n(1/e)\}_{n=1}^{\infty}$ is well-defined and has a unique limit $\overline{x}$, then we have $\log_y \overline{x} = \overline{x}$, i.e., $y = \overline{x}^{1/\overline{x}}$, and hence the limit of the sequence is located in the interval $(1/e, 1)$. As in the beginning part of the proof of (c), we obtain $f^2(1/e) < 1/e$ from $e^{-e} < y$ by way of $-1/\ln y > 1/e$, $f(1/e) > 1/e$, $|f'(c)| > 1$, and $f^2(1/e) - f(1/e) < (1/e) - f(1/e)$, applied in the listed order. Note that $f^2(x) = \log_y \log_y x$ is strictly increasing, so order-preserving, for x in the interval $(0, 1)$. Applying the order-preserving f^2 to $f^2(1/e) < 1/e$, we obtain

$$\cdots < f^6\left(\frac{1}{e}\right) < f^4\left(\frac{1}{e}\right) < f^2\left(\frac{1}{e}\right) < \frac{1}{e}.$$

The sequence $\{f^{2n}(1/e)\}_{n=1}^{\infty}$ also has the same limit $\overline{x}$, so we have

$$\overline{x} < \frac{1}{e} \quad \text{and} \quad \overline{x} > \frac{1}{e}.$$

From this contradiction, we conclude that the sequence $\{f^n(1/e)\}_{n=1}^{\infty}$ is not well-defined or diverges.

The remaining case $y = e^{-e}$ is trivial to verify. ■

3. INVERSE FUNCTIONS OF $y = x^x$ AND $y = xe^x$, AND FUNCTIONAL REPRESENTATIONS OF $w^w = x^x$ AND $x^z = z^x$. Let $x = h(y)$ denote the inverse multifunction of $y = x^{1/x}$, so that h is given by the three types of inverse of $y = x^{1/x}$ in Theorem 1. Then we can represent the inverse function of $y = x^x$ by using of h and a substitution $s = 1/x$. We see that $y = x^x = 1/s^{1/s}$, so $1/y = s^{1/s}$, hence $s = h(1/y)$. Therefore the inverse function of $y = x^x$ is $x = 1/h(1/y)$. We deduce a result for the inverse functions of $y = x^x$.

Corollary 2. *The inverse functions of $y = x^x$ are*

$$x = p(y) = \begin{cases} \left(\cdots \log_{\frac{1}{y}} \log_{\frac{1}{y}} \dfrac{1}{e}\right)^{-1}, & (x \geq e, y \geq e^e) \\ y^{\frac{1}{y}^{\frac{1}{y}^{\cdot^{\cdot^{\cdot}}}}}, & \left(\dfrac{1}{e} \leq x \leq e, e^{-\frac{1}{e}} \leq y \leq e^e\right) \\ \left(\cdots \log_{\frac{1}{y}} \log_{\frac{1}{y}} e\right)^{-1}, & \left(0 < x \leq \dfrac{1}{e}, e^{-\frac{1}{e}} \leq y < 1\right). \end{cases}$$

We know that the inverse multi-function of $y = x^x$ is $x = p(y)$. Then we can represent the inverse function of $y = xe^x$ by using p. We see that $y = xe^x$ converts to $e^y = (e^x)^{(e^x)}$, hence $e^x = p(e^y)$, so we have $x = \ln p(e^y)$. Therefore the inverse multi-function of $y = xe^x$ is $x = \ln p(e^y)$; this inverse function, called Lambert's W-function, is useful in many areas [**1**]. We conclude with a result for the inverse functions of $y = xe^x$.

Corollary 3. *The inverse functions of $y = xe^x$ are*

$$x = \begin{cases} -\log_e\left(\cdots\log_{(e^{-y})}\log_{(e^{-y})}\dfrac{1}{e}\right), & (x \geq 1, y \geq e) \\ y(e^{-y})^{(e^{-y})^{\cdot^{\cdot^{\cdot}}}}, & \left(-1 \leq x \leq 1, -\dfrac{1}{e} \leq y \leq e\right) \\ -\log_e(\cdots\log_{(e^{-y})}\log_{(e^{-y})} e), & \left(x \leq -1, -\dfrac{1}{e} \leq y < 0\right). \end{cases}$$

Also we can find the full representations of inverse functions of $y = x\ln x$, $y = x + \ln x$, $y = xe^{-x}$, $y = x - \ln x$, $y = x + e^x$, and so on, by similar methods. In particular, the inverse function of $y = x + e^x$ gives an explicit solution of the equation $e^x + x = 0$, namely

$$-\left(\frac{1}{e}\right)^{\left(\frac{1}{e}\right)^{\cdot^{\cdot^{\cdot}}}}.$$

Next we state the functional representations of $w^w = x^x$ and $x^z = z^x$. These are easily obtained, so we show only the results.

Corollary 4. *The functional representations of $w^w = x^x$ are $w = x$ and*

$$w = \begin{cases} (x^x)^{(x^{-x})^{(x^{-x})^{\cdot^{\cdot^{\cdot}}}}}, & x \in \left(0, \dfrac{1}{e}\right] \\ \left(\cdots\log_{(x^{-x})}\log_{(x^{-x})} e\right)^{-1}, & x \in \left[\dfrac{1}{e}, 1\right). \end{cases}$$

Corollary 5. *The functional representations of $z^{1/z} = x^{1/x}$ are $z = x$ and*

$$z = \begin{cases} \left(x^{\frac{1}{x}}\right)^{\left(x^{\frac{1}{x}}\right)^{\cdot^{\cdot^{\cdot}}}}, & x \in [e, \infty) \\ \cdots\log_{\left(x^{\frac{1}{x}}\right)}\log_{\left(x^{\frac{1}{x}}\right)} e, & x \in (1, e]. \end{cases}$$

By the symmetry of $w^w = x^x$, the two representations in Corollary 4 are inverse functions to each other, and similarly for $z^{1/z} = x^{1/x}$.

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University of Seoul, Seoul 130-743, Korea
yhcho@uoscc.uos.ac.kr

ITPlus, Inc., FKI bldg. 4F, 28-1 Yoido-dong, Yonugdeungpo-gu, Seoul, Korea
pkh@itplus.co.kr

On the Total Edge-Length of a Tetrahedron

Hiroshi Maehara

For a tetrahedron $ABCD$ in $\mathbf{R}^3$, let $L(ABCD)$ denote the sum of its edge-lengths. We prove the following:

Theorem 1. *Let R be the radius of the sphere with minimum volume that encloses a given tetrahedron $ABCD$. Then $6R < L(ABCD) \le 4\sqrt{6}R$, $L(ABCD) = 4\sqrt{6}R$ only when $ABCD$ is a regular tetrahedron, and $L(ABCD) = 6R$ only in a limit degenerate case such as $A \neq B = C = D$.*

Denote the perimeter of a triangle ABC by $L(ABC)$. A similar result for $L(ABC)$ can be obtained easily.

Lemma 1. *Let $A = (a, 0)$, $B = (-a, 0)$, $C = (x_0, y_0)$, $y_0 > 0$, be three points in $\mathbf{R}^2$. If a point $X = (x, y)$ lies on the circle determined by A, B, C, and if $y > y_0$ then $\overline{AC} + \overline{BC} < \overline{AX} + \overline{BX}$.*

Proof. X is exterior to the ellipse with foci A, B that passes through C. ∎

Corollary 1. *Let R be the radius of the circle with minimum area that encloses a given triangle ABC in $\mathbf{R}^2$. Then $4R < L(ABC) \le 3\sqrt{3}R$, $L(ABC) = 3\sqrt{3}R$ only when ABC is an equilateral triangle, and $L(ABC) = 4R$ only in a limit degenerate case.*

Proof. Let Γ be the circle with minimum area that encloses the triangle ABC in the plane, and let O be its center. (Γ might not be the circumscribed circle of the triangle ABC.) Then O lies either inside the triangle ABC or on a side of ABC. First, consider the case that O lies on a side, say, on the side AB. Then $\overline{AB} = 2R$ and $\overline{AC} + \overline{CB} > \overline{AB}$. Hence $L(ABC) > 4R$. Next, suppose that O is interior to the triangle ABC. We may put $A = (a, 0)$, $B = (-a, 0)$, $C = (x_1, y_1)$, $y_1 > 0$. Let AC' be a diameter of Γ, and let $C' = (x_0, y_0)$. Then, since O is interior to ΔABC, it follows that $0 < y_0 < y_1$. Hence Lemma 1 ensures that $L(ABC) > L(ABC')$. Since O lies on AC', it follows that $L(ABC') > 4R$. Hence $L(ABC) > 4R$.