A Mechanical Derivation of the Area of the Sphere

David Garber and Boaz Tsaban

1. INTRODUCTION. In the beginning of the twelfth century CE, an interesting new geometry book appeared: *The Book of Mensuration of the Earth and its Division*, by Rabbi Abraham Bar Hiya (acronym RABH), a Jewish philosopher and scientist. This book is interesting both historically and mathematically. Its historical aspect is discussed in [**3**]. In this paper we consider the mathematical aspect.

The second part of the book contains a beautiful mechanical derivation of the area of the disk [**5**, §95]. Roughly speaking, the argument goes as follows (see Figure 1): The disk is viewed as the collection of all the concentric circles it contains. If we cut

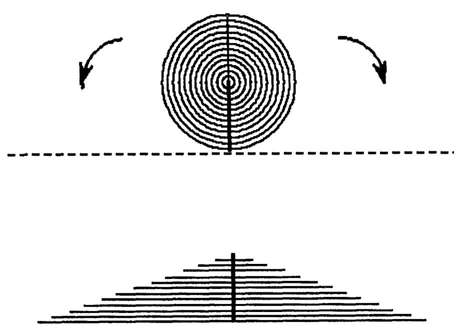

Figure 1.

the circles along the radius of the disk, and let them fan out to become straight lines, we get a triangle (because the ratio of the circumferences of the circles to their diameters is constant). The base length of the resulting triangle is equal to the circumference of the original circle, and its height is equal to the radius of this circle. Thus, the area of a circle is equal to half of the product of the radius and the circumference. Using modern terms, this means that the area of the disk with radius R is equal to

$$\frac{2\pi R \cdot R}{2} = \pi R^2.$$

This is not the first derivation of the area of the disk from its circumference. Archimedes preceded RABH by more than thirteen centuries. The advantage of the Archimedean proof over RABH's is its mathematical completeness. However, the novelty of RABH's proof lies in the fact that it can be grasped by the reader who does not have solid mathematical knowledge.

2. A MODERN PROOF OF THE VALIDITY OF RABH'S ARGUMENT. Criticism of this proof is discussed in [**1**], [**2**], and [**3**]. The proof, however, remained mathematically incomplete until 1991, when we used modern mathematical tools to justify it. Our formalization appears in [**4**]. We give here a brief description.

The modern method of obtaining areas from lengths is the integral calculus. Advanced calculus gives us a strong tool for finding the change in the area of a given shape under continuously differentiable transformations—namely, the Jacobian. We recall the following well known fact:

Fact 1. *Let A and B be two planar figures. If there is a continuously differentiable bijection $g : A \to B$ satisfying $Jg(x) \neq 0$ for all x in A, then the area of B is equal to $\int_A |Jg|$, where Jg denotes the determinant of the differential matrix of g. In particular, if $Jg \equiv 1$, then the areas of A and B are equal.*

RABH's proof defines a bijection between a disk and a triangle. For convenience, we take g to be the inverse of RABH's bijection. That is, g maps each line segment from the triangle to a corresponding concentric circle; see Figure 2:

$$g(r, u) = \left(r \cos \frac{u}{r}, r \sin \frac{u}{r}\right) \qquad (0 < r \leq R; -\pi r < u < \pi r)$$

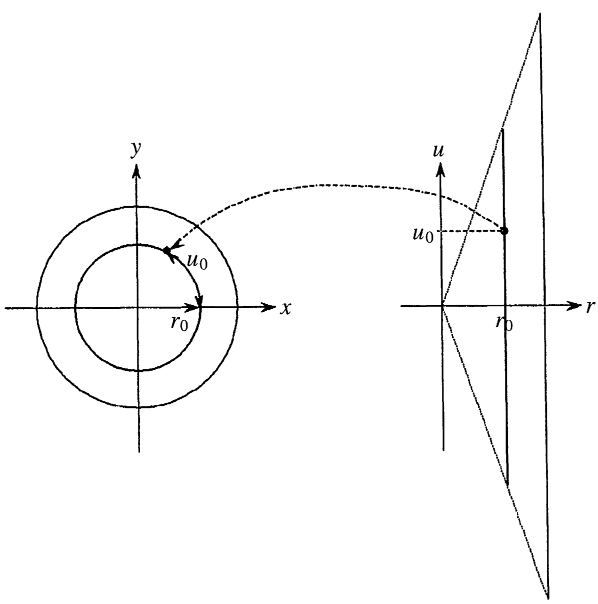

Figure 2.

Note that g is an injection, as r is the radius of the concentric circle to which the image $g(r, u)$ belongs, and u is the (signed) length measured on the arc of this circle, from the positive part of the x axis.

We now check that the Jacobian of this transformation is 1:

$$Jg(r, u) = \det \begin{bmatrix} \dfrac{\partial \left(r \cos \frac{u}{r}\right)}{\partial r} & \dfrac{\partial \left(r \cos \frac{u}{r}\right)}{\partial u} \\ \dfrac{\partial \left(r \sin \frac{u}{r}\right)}{\partial r} & \dfrac{\partial \left(r \sin \frac{u}{r}\right)}{\partial u} \end{bmatrix} =$$

$$= \det \begin{bmatrix} \cos\frac{u}{r} + r\left(-\sin\frac{u}{r}\right)\frac{-u}{r^2} & r\left(-\sin\frac{u}{r}\right)\frac{1}{r} \\ \sin\frac{u}{r} + r\left(\cos\frac{u}{r}\right)\frac{-u}{r^2} & r\left(\cos\frac{u}{r}\right)\frac{1}{r} \end{bmatrix} =$$

$$= \det \begin{bmatrix} \cos\frac{u}{r} + \frac{u}{r}\sin\frac{u}{r} & -\sin\frac{u}{r} \\ \sin\frac{u}{r} + \frac{-u}{r}\cos\frac{u}{r} & \cos\frac{u}{r} \end{bmatrix} =$$

$$= \left(\cos\frac{u}{r} + \frac{u}{r}\sin\frac{u}{r}\right)\cos\frac{u}{r} + \sin\frac{u}{r}\left(\sin\frac{u}{r} + \frac{-u}{r}\cos\frac{u}{r}\right) =$$

$$= \cos^2\frac{u}{r} + \frac{u}{r}\sin\frac{u}{r}\cos\frac{u}{r} + \sin^2\frac{u}{r} - \frac{u}{r}\sin\frac{u}{r}\cos\frac{u}{r} =$$

$$= \cos^2\frac{u}{r} + \sin^2\frac{u}{r} = 1.$$

3. DERIVING THE AREA OF THE SPHERE. We now use RABH's method to derive the surface area of the sphere. We view the sphere as consisting of the horizontal circles contained in it. First, we cut the circles along the dotted part of the boldfaced large circle (see Figure 3) and straighten them. The resulting surface is bent along the second half of the large circle (Figure 4).

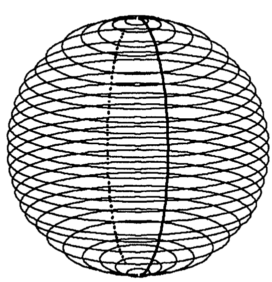

Figure 3.

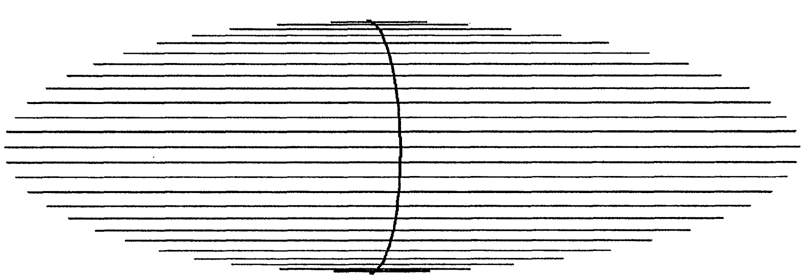

Figure 4.

Now we can straighten this bent surface to yield a planar figure (Figure 5). From symmetry of the right and left parts of Figure 5, it is enough to find the area of the right half (which corresponds to the right half of the sphere). The area of this part can be found once we have an explicit description of its graph. In Figure 6, the variable u ranges from $-\pi R/2$ to $\pi R/2$, and for each value of u, v ranges from 0 to πr, where $r = R\cos(u/R)$. Therefore, the graph is described by the function $v = \pi R\cos(u/R)$.

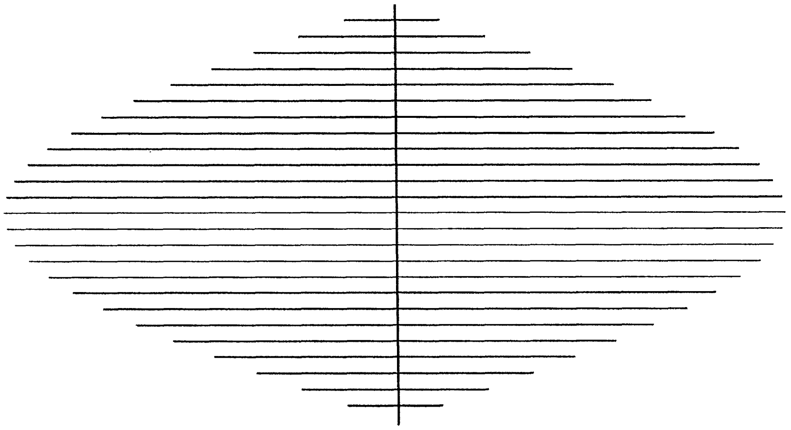

Figure 5.

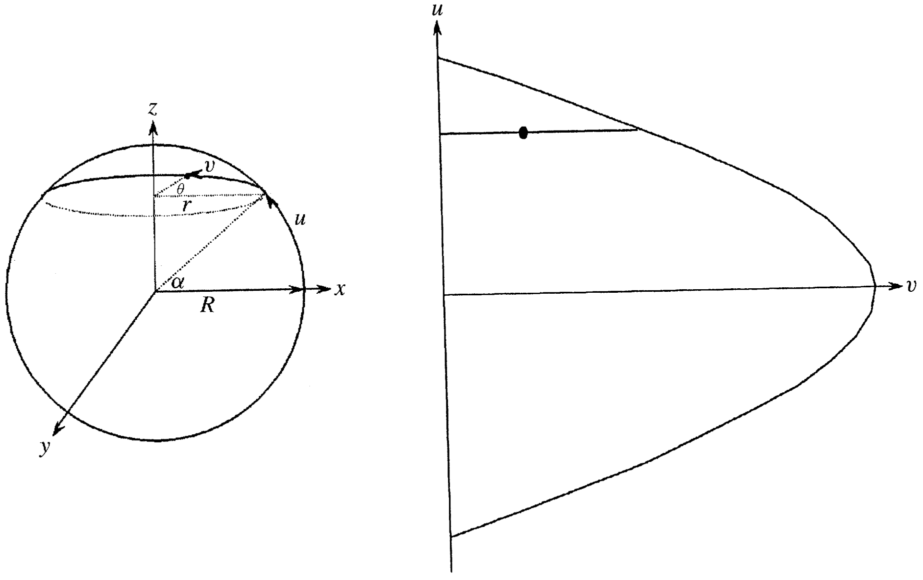

Figure 6.

We can now find the area:

$$\int_{-\pi R/2}^{\pi R/2} \pi R\cos(u/R)du = \pi R\,[R\sin(u/R)]_{-\pi R/2}^{\pi R/2} = 2\pi R^2. \tag{1}$$

This is the area of half a sphere, so the area of a sphere is $4\pi R^2$.

Calculation of the integral in (1) can actually be avoided: Since the area bounded by the cosine graph and the x axis in the range $[-\pi/2, \pi/2]$ is equal to 2 ($= \int_{-\pi/2}^{\pi/2} \cos x\, dx$), the area below $\cos(x/R)$ in the range $[-\pi R/2, \pi R/2]$ is equal to

$2R$. As $v = \pi R\cos(u/R)$, the desired area is $\pi R \cdot 2R = 2\pi R^2$. Thus, the whole argument can be viewed as "mechanical". However, such an argument could not exist in RABH's time, since the notions of a graph (and an area below a graph) were not developed then.

4. FORMALIZING THE DERIVATION. We now formalize our argument. Let $g : \mathbb{R}^2 \to \mathbb{R}^3$ be the transformation that assigns to each point (u, v) in the planar figure the corresponding point (x, y, z) on the sphere.

Write $g(u, v) := (x(u, v), y(u, v), z(u, v))$. Then the area of the sphere is equal to $\int \|\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\| du\, dv$, where $\times$ is the vector product:

$$\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v} \stackrel{\text{def}}{=} \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{bmatrix}. \tag{2}$$

Let $\alpha = u/R$ be the angle corresponding to the arc u on the sphere, and let $\theta = v/(R\cos\alpha)$ be the angle corresponding to the arc v on the sphere (see Figure 6). The coordinates of the point at the end of v are:

- $x = R\cos\alpha\cos\theta$ (thus $\frac{\partial x}{\partial u} = -\sin\alpha\cos\theta - \theta\sin\theta\sin\alpha$, and $\frac{\partial x}{\partial v} = -\sin\theta$),
- $y = R\cos\alpha\sin\theta$ (thus $\frac{\partial y}{\partial u} = -\sin\alpha\sin\theta + \theta\cos\theta\sin\alpha$, and $\frac{\partial y}{\partial v} = \cos\theta$),
- $z = R\sin\alpha$ (thus $\frac{\partial z}{\partial u} = \cos\alpha$, and $\frac{\partial z}{\partial v} = 0$).

The vector product (2) is

$$(-\cos\theta\cos\alpha, -\cos\alpha\sin\theta, -\sin\alpha),$$

so

$$\left\|\frac{\partial g}{\partial u} \times \frac{\partial g}{\partial v}\right\| = \sqrt{\cos^2\alpha\cos^2\theta + \cos^2\alpha\sin^2\theta + \sin^2\alpha} = 1.$$

5. THE AREA OF HORIZONTAL SECTIONS OF THE SPHERE. Applying RABH's method to horizontal sections of the sphere is now straightforward: We simply let u vary in the appropriate range (see Figure 6). Consider the upper half of the sphere. The area of the horizontal section of height $h = \alpha R$ is $2\int_0^{\alpha R} \pi R\cos(u/R)du = 2\pi R\,[R\sin(u/R)]_0^{\alpha R} = 2\pi R(R\sin\alpha) = 2\pi Rh$. This implies that the area of the horizontal section starting at height a and ending at height $b = a + \Delta$ is

$$2\pi Rb - 2\pi Ra = 2\pi R\Delta.$$

Thus, we have a "mechanical" explanation of the interesting fact that the section's area depends only on Δ.

6. REVERSING THE USUAL COMPUTATIONS. It is common to derive the area of the sphere from the volume of the corresponding ball using the derivation

$$\frac{d}{dR}\left(\frac{4}{3}\pi R^3\right) = 4\pi R^2.$$

However, it takes a lot of effort to explain why this derivative is equal to the area of the sphere.

Using the ideas from the earlier sections, we can reverse the computation and find the volume of the ball by summing up the areas of the concentric spheres that are contained in it:

$$\int_0^R 4\pi r^2 dr = \frac{4}{3}\pi R^3.$$

We conclude with the curiosity that mechanically, this derivation of the volume of the ball corresponds to flattening all of the spheres as in Section 3, and then calculating the volume of the resulting shape (see Figure 7).

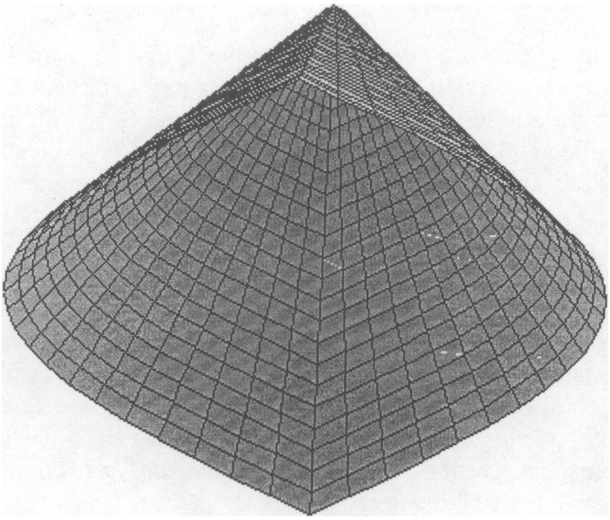

Figure 7.

REFERENCES

1. Michael Bleicher, The solution of Havot-Yaiir's Problem on the measurement of the area of a circle, *Spices for Wisdom* **1** (1984) 79–87. [Hebrew]
2. David Garber and Boaz Tsaban, A Circular Booth II, *Magal* **11** (1995) 127–134. [Hebrew]
3. ———, The Proof of Rabbi Abraham Bar Hiya Hanasi, in preparation.
4. ———, Every [Circle] whose Circumference, *Higayon* **3** (1995) 103–131. [Hebrew]
5. Yehiel Michael HaCohen Guttmann, *Chibbur Ha-Meschicha WeHa-Tischboreth*, Schriften des Vereins Mekize Nirdamim, Berlin, 1913.

DAVID GARBER AND BOAZ TSABAN are Ph.D. students and teaching assistants in the Department of Mathematics and Computer Science at Bar-Ilan University. Their M.Sc. theses are in the fields of algebraic geometry and modern set theory, respectively. In addition to their publications in pure mathematics, they have published together more than ten papers on mathematics in Jewish texts. Their work has won the Brechfeld Prize for studies in Rabbinical logic, and they are working to establish a research center for mathematics in Jewish texts.

Department of Mathematics and Computer Science, Bar-Ilan University, 52900 Ramat-Gan, Israel
garber@macs.biu.ac.il, tsaban@macs.biu.ac.il