Tame Parahoric Nonabelian Hodge Correspondence

(joint work with P. Huang, G. Kydonakis and L. Zhao)

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Content

Introduction

Dolbeault, de Rham, Betti moduli spaces

Let X be a smooth projective variety over $\mathbb{C}$.

- $\mathcal{M}_{\mathrm{Dol}}$: the moduli space of stable Higgs bundles (E, ϕ) with vanishing Chern classes
- $\mathcal{M}_{\mathrm{dR}}$: the moduli space of stable vector bundles with integrable connections (V, ∇)
- $\mathcal{M}_{\mathrm{B}}$: the moduli space of irreducible representations $\mathcal{M}_{\mathrm{B}} = \mathrm{Hom}(\pi_1(X), \mathrm{GL}_n(\mathbb{C}))//G$

Corlette-Simpson correspondence (Simpson, 1994)

We have an isomorphism of set of points

$$\mathcal{M}_{\mathrm{Dol}} \cong \mathcal{M}_{\mathrm{dR}} \cong \mathcal{M}_{\mathrm{B}}.$$

The first isomorphism can be improved to be an homeomorphism of topological spaces and the second isomorphism can be improved to be an isomorphism of complex analytic spaces.

Remark

The first isomorphism is established with the help of harmonic (metrics) bundles. The second isomorphism is induced from the Riemann-Hilbert correspondence.

Estabilishing the Correspondence

1. Existence of harmonic metric
2. Riemann-Hilbert correspondence
3. correspondence among sets (categories)
4. correspondence among moduli spaces: analytic and algebraic

Noncompact Case

Big Pictures

Let X be a smooth projective variety with a reduced effective divisor (normal crossing) D. Establish the correspondence on $X \backslash \boldsymbol{D}$.

- Base: curves vs. higher dimension varieties
- Order of poles: $= 1$ (tame, regular singular) or ≥ 2 (wild, irregular singular)
- Structure group: $\mathrm{GL}_n(\mathbb{C})$ or G

Literature Review

1. Curves, tame and $\mathrm{GL}_n(\mathbb{C})$: Simpson (1990)
2. Curves, wild and $\mathrm{GL}_n(\mathbb{C})$: Sabbah (1999), Biquard-Boalch (2004)
3. Higher dimension, wild and $\mathrm{GL}_n(\mathbb{C})$: Mochizuki (2010)
4. Curves, wild and G: Boalch (2014)
5. Curves, tame and G: Biquard, Garcia-Prada, Mundet i Riera (2020)

Case: Curves, tame and $\mathrm{GL}_n(\mathbb{C})$

Parabolic Bundles

A parabolic bundle $E_\bullet$ on $(X, \boldsymbol{D})$ is a bundle E of rank n such that for each puncture $x \in \boldsymbol{D}$, we have a weighted filtration

$$\begin{aligned} E_x =& E_{x1} \supseteq \cdots \supseteq E_{xr} \supseteq E_{x,r+1} = \{0\} \\ 0 \leq& \alpha_1 < \cdots < \alpha_r < 1, \end{aligned}$$

where α_i are rational (or real) numbers.

Parabolic Higgs Bundles

A parabolic Higgs bundle is a pair $(E_\bullet, \phi)$, where $E_\bullet$ is a parabolic bundle and $\phi : E \to E \otimes K_X(\boldsymbol{D})$ is a morphism preserving the parabolic structure of $E_\bullet$, i,e, $\phi|_x : E_{xi} \to E_{xi} \otimes K_X(\boldsymbol{D})$.

Parabolic Degree

Let $E_\bullet$ be a parabolic bundle. The parabolic degree of $E_\bullet$ is

$$par\deg E_\bullet = \deg E + \sum_{x\in \boldsymbol{D}} \sum_i \alpha_i \dim(E_{xi}/E_{x,i+1}).$$

Stability Condition

A parabolic bundle $E_\bullet$ is semistable (resp. stable), if for any parabolic subbundle $F_\bullet \subseteq E_\bullet$, we have

$$\frac{par\deg E_\bullet}{\text{rk } E} \le \frac{par\deg F_\bullet}{\text{rk } F} \quad (\text{resp. } <)$$

Remark

The stability condition for parabolic Higgs bundles and parabolic regular D_X-modules can be defined similarly. The stability condition for filtered local systems is given by the part of "weights".

Simpson's Result

Three Categories

Let X be a smooth algebraic curve over $\mathbb{C}$ with a fixed reduced effective divisor $\boldsymbol{D}$. Denote by $X_{\boldsymbol{D}} := X \backslash \boldsymbol{D}$ the punctured curves.

- $\mathcal{M}_{\mathrm{Dol}}$: stable parabolic (filtered) Higgs bundles (E, ϕ) on X with parabolic degree zero
- $\mathcal{M}_{\mathrm{dR}}$: stable parabolic (filtered) regular D_X-modules (V, ∇) on X of parabolic degree zero
- $\mathcal{M}_{\mathrm{B}}$: stable filtered local systems of degree zero $\mathrm{Hom}^s(\pi_1(X_{\boldsymbol{D}}), \mathrm{GL}_n(\mathbb{C}))/\mathrm{GL}_n(\mathbb{C})$

Tame Nonabelian Hodge Correspondence on Noncompact Curves (Simpson 1990)

The above three categories are equivalent.

Local Data

Fix a point $x \in \boldsymbol{D}$. For line bundles, we have the following table of local data:

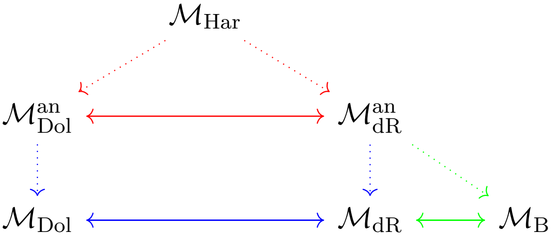
	Dolbeault	de Rham	Betti
weights	α	$\alpha - (s_\alpha + \bar{s}_\alpha)$	$-2(s_\alpha + \bar{s}_\alpha)$
eigenvalues	s_α	$\alpha + (s_\alpha - \bar{s}_\alpha)$	$\exp(-2\pi i(\alpha + (s_\alpha - \bar{s}_\alpha)))$

Tameness

Let (E, ∂''_E) be a holomorphic bundle on X_D with a Higgs field $\phi : E \to E \otimes \Omega^1_{X_D}$. The Higgs field ϕ is *tame* if the eigenvalues of ϕ have poles of order at most one.

Three Extra Categories

- $\mathcal{M}^{\mathrm{an}}_{\mathrm{Dol}}$: stable tame (analytic) metrized Higgs bundles $(E, \partial''_E, h, \phi)$ on X_D with (analytic) degree zero
- $\mathcal{M}^{\mathrm{an}}_{\mathrm{dR}}$: stable regular (analytic) metrized D_X-modules (V, h, ∇) on X_D of (analytic) degree zero
- $\mathcal{M}_{\mathrm{Har}}$: tame harmonic bundles on X_D



Remark

1. Red: analytic stable tame Higgs bundle = analyic stable regular D_X-modules on $X_{\boldsymbol{D}}$
2. Blue: analytic objects $\longrightarrow$ algebraic objects
3. Green: a version of Riemann-Hilbert correspondence

Why not Parabolic

Example 1

Consider $B(z) = \begin{pmatrix} 0 & z \\ z^{-1} & 0 \end{pmatrix}$. Denote by

$$\phi(z) = B(z)\frac{dz}{z} = \left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \frac{1}{z^2} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right) dz$$

the corresponding Higgs field.

Example 1 (continued)

Take $g = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix}$, we have

$$\mathrm{Ad}(g)\phi(z) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{dz}{z},$$

which can be taken as a Higgs field for parabolic bundles.

SL_2-case

Clearly, $B(z) \in \mathfrak{sl}_2(\mathbb{C}((z)))$. We claim that as a $\mathfrak{sl}_2$ matrix, $\phi(z)$ does not conjugate to $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \frac{dz}{z}$ (or any $B'(z)\frac{dz}{z}$, where $B'(z) \in \mathfrak{sl}_2(\mathbb{C}[[z]])\frac{dz}{z}$).
This statement comes from Bruhat-Tits theory.

Example 2

Let $A(z) = \sum_{i=0} a_i z^i$ be an element in $\mathfrak{gl}_n(\mathbb{C}[[z]])$, and we consider $A(z)\frac{dz}{z}$ as a connection form with regular singularities. The the gauge action of $g \in \mathrm{GL}_n(\mathbb{C}((z)))$ on $A(z)\frac{dz}{z}$ is defined as

$$g \circ A(z)\frac{dz}{z} := (\mathrm{Ad}(g)A(z))\,\frac{dz}{z} + dg \cdot g^{-1}.$$

There exists an element g such that

$$g \circ A(z)\frac{dz}{z} = a\frac{dz}{z},$$

where $a \in \mathfrak{gl}_n(\mathbb{C})$ is a constant matrix. The monodromy around the puncture is exactly $\exp(-2\pi\sqrt{-1}a)$,

Example 2 (Continued): SL_2

Consider

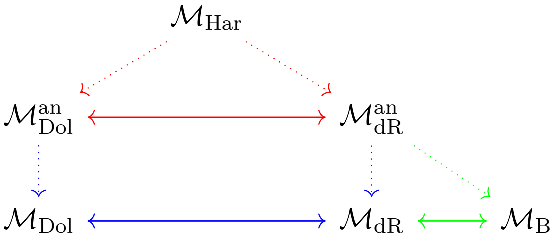
$$A(z) = \begin{pmatrix} \frac{m}{2} & 0 \\ 0 & -\frac{m}{2} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} z^m,$$

as an element in $\mathfrak{sl}_2(\mathbb{C}[[z]])$. Then, the connection form $A(z)\frac{dz}{z}$ is gauge equivalent to $a\frac{dz}{z}$ for some $a \in \mathfrak{sl}_2(\mathbb{C})$ if and only if m is even (Babbitt, Varadarajan, 1983). For a general complex reductive group G, the connection form $A(z)\frac{dz}{z}$ is not gauge equivalent to the form $a\frac{dz}{z}$ in general.

Remark

Boalch (2011) introduce the parahoric objects to establish the correspondence between equivalence classes of G-connection forms and equivalence classes of monodromies around punctures.

Tame Parahoric Nonabelian Hodge Correspondence (Big Picture)



Remark

1. Green: tame parahoric Riemann-Hilbert correspondence (Boalch, 2011)
2. Red: analytic stable tame G-Higgs bundle = analytic stable regular (D_X, G)-modules on $X_{\boldsymbol{D}}$
3. Blue: analytic objects $\longrightarrow$ algebraic objects (parahoric objects)
4. Green: add stability condition to Riemann-Hilbert

Parahoric Group Scheme (local)

Let G be a connected complex reductive group. We fix a maximal torus T in G with Lie algebras $\mathfrak{t}$ and $\mathfrak{g}$. Let θ be an element in $\mathrm{Hom}(\mathbb{C}^*, T) \otimes_{\mathbb{Z}} \mathbb{Q}$, which is called *a rational weight* (can be considered as an element in $\mathfrak{t}$ with rational coefficients). Let $R := \mathbb{C}[[z]]$ and $K := \mathbb{C}((z))$. We define the *parahoric subgroup* $G_\theta(K)$ of $G(K)$ as

$$G_\theta(K) := \langle T(R), U_r(z^{m_r(\theta)}R), r \in \mathcal{R} \rangle,$$

where

$$m_r(\theta) := \lceil -r(\theta) \rceil,$$

Denote by $\mathcal{G}_\theta$ the corresponding group scheme of $G_\theta(K)$, which is called the *parahoric group scheme*.

Example

Consider $G = \mathrm{SL}_2(\mathbb{C}) \subseteq \mathrm{GL}_2(\mathbb{C})$. Let $\theta = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$. Then, the parahoric subgroup $G_\theta(K)$ is

$$\begin{pmatrix} R & z^{-1}R \\ zR & R \end{pmatrix}.$$

Parahoric Group Scheme (global)

Let $\boldsymbol{\theta} = \{\theta_x, x \in \boldsymbol{D}\}$ be a collection of weights over points in $\boldsymbol{D}$, for a curve X and a group G as above. We define a group scheme $\mathcal{G}_{\boldsymbol{\theta}}$ over X by gluing the following local data

$$\mathcal{G}_{\boldsymbol{\theta}}|_{X_{\boldsymbol{D}}} \cong G \times X_{\boldsymbol{D}}, \quad \mathcal{G}_{\boldsymbol{\theta}}|_{\mathbb{D}_x} \cong \mathcal{G}_{\theta_x}, x \in \boldsymbol{D},$$

where $\mathbb{D}_x$ is a formal disc around x. This group scheme $\mathcal{G}_{\boldsymbol{\theta}}$ will be called a *parahoric (Bruhat–Tits) group scheme*.

Tame Parahoric Higgs Torsors

A *tame parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-Higgs torsor* on a smooth algebraic curve X is a pair (E, φ), where

- E is a $\mathcal{G}_{\boldsymbol{\theta}}$-torsor on X;
- $\varphi \in H^0(X, \mathrm{Ad}(E) \otimes K_X(\boldsymbol{D}))$ is a section.

The section φ is called a *tame parahoric Higgs field*.

Logahoric D_X-modules

A *logahoric* $(D_X, \mathcal{G}_{\boldsymbol{\theta}})$*-module* on X is a pair (E, ∇), where E is a parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsor and $\nabla : \mathcal{O}_E \to \mathcal{O}_E \otimes K_X(D)$ is connection, which is called a *logahoric $\mathcal{G}_{\boldsymbol{\theta}}$-connection* on E.

Line bundles

Let P be a parabolic subgroup of G. Define $\mathcal{P}_{\boldsymbol{\theta}} \subseteq \mathcal{G}_{\boldsymbol{\theta}}$ a subgroup scheme. Let $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$ be a reduction of the structure group.

$$\begin{array}{ccc} E_\varsigma & \dashrightarrow & E \\ \vdots & & \downarrow \\ X & \xrightarrow{\varsigma} & E/\mathcal{P}_{\boldsymbol{\theta}} \end{array}$$

Let $\kappa : \mathcal{P}_{\boldsymbol{\theta}} \to \mathbb{G}_m$ be a morphism of group schemes over X. Define the line bundle $L(\kappa, \varsigma) := \kappa_* E_\varsigma$.

Parahoric Degree

We define the *parahoric degree* of a $\mathcal{G}_{\boldsymbol{\theta}}$-torsor E with respect to a given reduction ς and a character κ as follows

$$parh \deg E(\varsigma, \kappa) = \deg(L(\kappa, \varsigma)) + \langle \boldsymbol{\theta}, \kappa \rangle,$$

where $\langle \boldsymbol{\theta}, \kappa \rangle := \sum_{x \in D} \langle \theta_x, \kappa \rangle$.

Stability Conditions

A parahoric $\mathcal{G}_{\boldsymbol{\theta}}$-torsor E is called *R-stable* (resp. *R-semistable*), if for

- any proper parabolic group $P \subseteq G$,
- any reduction of structure group $\varsigma : X \to E/\mathcal{P}_{\boldsymbol{\theta}}$,
- any nontrivial anti-dominant character $\kappa : \mathcal{P}_{\boldsymbol{\theta}} \to \mathbb{G}_m$, which is trivial on the center of $\mathcal{P}_{\boldsymbol{\theta}}$,

one has

$$parh\deg E(\varsigma, \kappa) > 0, \quad (\text{resp. } \geq 0).$$

Remark

A similar definition can be defined for parahoric Higgs torsors and logahoric D_X-modules.

Local Data

	Dolbeault	de Rham	Betti
weights	α	β	γ
eigenvalues	φ_α	∇_β	M_γ

- $\varphi_\alpha = s_\alpha + Y_\alpha$
- $\beta = \alpha - (s_\alpha + \bar{s}_\alpha)$, $\gamma = -(s_\alpha + \bar{s}_\alpha)$
- $\nabla_\beta = \alpha + (s_\alpha - \bar{s}_\alpha) - (H_\alpha + X_\alpha - Y_\alpha)$
- $M_\gamma = \exp\left(-2\pi i(\alpha + s_\alpha - \bar{s}_\alpha)\right) \exp\left(2\pi i(H_\alpha + X_\alpha - Y_\alpha)\right).$

Three Categories (Moduli Spaces)

- $\mathcal{C}_{\mathrm{Dol}}(X, \mathcal{G}_{\boldsymbol{\alpha}}, \varphi_{\boldsymbol{\alpha}})$: R-stable tame parahoric $\mathcal{G}_{\boldsymbol{\alpha}}$-Higgs torsors on X with degree zero and residues $\varphi_{\boldsymbol{\alpha}}$
- $\mathcal{C}_{\mathrm{dR}}(X, \mathcal{G}_{\boldsymbol{\beta}}, \nabla_{\boldsymbol{\beta}})$: R-stable tame logahoric $(D_X, \mathcal{G}_{\boldsymbol{\beta}})$-modules on X with parahoric degree zero and residues $\nabla_{\boldsymbol{\beta}}$
- $\mathcal{C}_{\mathrm{B}}(X_{\boldsymbol{D}}, G, \gamma, M_\gamma)$: stable γ-filtered G-local systems of degree zero with monodromies M_γ around punctures.

Notation

The letter $\mathcal{C}$ is for categories, and the letter $\mathcal{M}$ is for moduli spaces.

Main Results

Theorem (HKSZ, 2022)

The categories are equivalent

$$\mathcal{C}_{\mathrm{Dol}}(X, \mathcal{G}_{\boldsymbol{\alpha}}, \varphi_{\boldsymbol{\alpha}}) \cong \mathcal{C}_{\mathrm{dR}}(X, \mathcal{G}_{\boldsymbol{\beta}}, \nabla_{\boldsymbol{\beta}}) \cong \mathcal{C}_{\mathrm{B}}(X_{\boldsymbol{D}}, G, \boldsymbol{\gamma}, M_{\boldsymbol{\gamma}}).$$

Theorem (HKSZ, 2022)

There is an isomorphism of complex analytic spaces

$$\mathcal{M}_{\mathrm{B}}^{(\mathrm{an})}(X_{\boldsymbol{D}}, G, \boldsymbol{\gamma}, M_{\boldsymbol{\gamma}}) \cong \mathcal{M}_{\mathrm{dR}}^{(\mathrm{an})}(X, \mathcal{G}_{\boldsymbol{\beta}}, \nabla_{\boldsymbol{\beta}}),$$

and we also have a homeomorphism of topological spaces

$$\mathcal{M}_{\mathrm{Dol}}^{(\mathrm{top})}(X, \mathcal{G}_{\boldsymbol{\alpha}}, \varphi_{\boldsymbol{\alpha}}) \cong \mathcal{M}_{\mathrm{dR}}^{(\mathrm{top})}(X, \mathcal{G}_{\boldsymbol{\beta}}, \nabla_{\boldsymbol{\beta}}).$$

Reference

1. Tame Parahoric Higgs Torsors for a Complex Reductive Group, arXiv: 2107.01977 (with G. Kydonakis, L. Zhao)
2. Tame parahoric nonabelian Hodge correspondence on curves, arXiv: 2205.15475 (with P. Huang , G. Kydonakis, L. Zhao)
3. Tame Parahoric Nonabelian Hodge Correspondence in Positive Characteristic over Algebraic Curves, arXiv: 2109.00850 (with M. Li)

Thanks